DISTRIBUTION OF RADIANT THERMAL FLUX OVER THE SURFACE OF A SPHERE FOR HYPERSONIC FLOW OF A NONVISCOUS RADIATING GAS PAST THE SPHERE

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In the present study using the Newtonian approximation [1] we obtain an analytical solution to the problem of flow of a steady, uniform, hypersonic, nonviscous, radiating gas past a sphere. The three-dimensional radiative-loss approximation is used. A distribution is found for the gasdynamic parameters in the shock layer, the withdrawal of the shock wave and the radiant thermal flux to the surface of the sphere. The Newtonian approximation was used earlier in [2, 3] to analyze a gas flow with radiation near the critical line. In [2] the radiation field was considered in the differential approximation, with the optical absorption coefficient being assumed constant. In [3] the integrodifferential energy equation with account of radiation was solved numerically for a gray gas. In [4-7] the problem of the flow of a nonviscous, nonheat-conducting gas behind a shock wave with account of radiation was solved numerically. To calculate the radiation field in [4, 7] the three-dimensional radiative-loss approximation was used; in [5, 6] the self-absorption of the gas was taken into account. A comparison of the equations obtained in the present study for radiant flow from radiating air to a sphere with the numerical calculations [4-7] shows them to have satisfactory accuracy.

1. We consider the hypersonic flow of a nonviscous, nonheat-conducting radiating gas past a spherical body of radius R. The system of equations that describes the gas flow between the withdrawing shock wave and the sphere, written in a spherical coordinate system fixed in the body, is given in [5]. This system of equations is solved with boundary conditions on the oblique discontinuity and with the condition that there is no flow on the body. It is further assumed that the gas is ideal, the shock layer is thin, and the conditions of the hypersonic approximation are satisfied, so that

$$p_\infty \ll \rho_\infty V_\infty^2, \qquad h_\infty \ll V_\infty^2/2$$

where p_∞ is pressure, ρ_∞ is density, h_∞ is enthalpy, and V_∞ is velocity of the incident flow.

The gas satisfies the equation of state

$$h = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} \tag{1.1}$$

where γ is the effective ratio of specific heats behind a compaction discontinuity, p is pressure, h is enthalpy, and ρ is density.

To solve the problem we expand quantities in the small parameter ε, equal to the ratio of the gas density before and after the shock wave (Newtonian approximation) [1]

$$\begin{gathered}(r-R)/R = \varepsilon y_0 + \ldots, \qquad u = V_\infty[u_0 + O(\varepsilon)], \qquad v = V_\infty[\varepsilon v_0 + O(\varepsilon^2)] \\ p = \rho_\infty V_\infty^2[p_0 + O(\varepsilon)], \qquad \rho = \frac{\rho_\infty}{\varepsilon}[\rho_0 + O(\varepsilon)], \qquad h = \frac{V_\infty^2}{2}[h_0 + O(\varepsilon)] \\ Q = \frac{\sigma}{\varepsilon R}\left(\frac{V_\infty^2}{2C_p}\right)^4[Q_0 + O(\varepsilon)]\end{gathered} \tag{1.2}$$

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Here r is radial coordinate, u is tangential velocity component, v is normal velocity component, p is pressure, h is enthalpy, ρ is density, σ is the Stefan–Boltzmann constant, R is the radius of the body, $Q = \operatorname{div} q_R$ is the divergence of the radiant flow, $\varepsilon = (\gamma - 1)/(\gamma + 1)$, and C_p is the effective specific heat at constant pressure.

The subscript ∞ indicates quantities in the incident flow, the subscript S indicates quantities on the shock wave, and the subscript 0 indicates dimensionless quantities.

Equations (1.2) are substituted into the equations of gasdynamics, after which we obtain for the first terms of the expansion (the subscript 0 is dropped)

$$\frac{\partial}{\partial y}\rho v + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\rho u) = 0, \qquad \rho u^2 = j\frac{\partial p}{\partial y}$$
$$\rho v\frac{\partial u}{\partial y} + \rho u\frac{\partial u}{\partial\theta} = 0, \qquad \rho v\frac{\partial h}{\partial y} + \rho u\frac{\partial h}{\partial\theta} = -\Gamma Q \tag{1.3}$$
$$h = \frac{2\gamma}{\gamma - 1}\frac{p}{\rho}, \qquad \Gamma = \frac{2\sigma}{\rho_\infty V_\infty^3}\left(\frac{V_\infty^2}{2C_p}\right)^4$$

Here Γ is the radiation parameter [5] and θ is the angular coordinate.

To the same approximation the relations on the shock wave take the form

$$u_S - \sin\theta, \qquad v_S = -\cos\theta, \qquad p_S = \cos^2\theta, \qquad h_S = \cos^2\theta \tag{1.4}$$

The condition of nonflow of the body takes the form

$$v\,(y = 0) = 0 \tag{1.5}$$

We determine the dimensionless stream function

$$d\psi = \rho u \sin\theta\, dy - \rho v \sin\theta\, d\theta \tag{1.6}$$

Further, in system (1.3) we transform to the variables $\Psi = \theta$. Then in these variables the system (1.3) is written as follows:

$$\partial u/\partial\theta = 0 \tag{1.7}$$

$$\partial p/\partial\Psi = u/\sin\theta \tag{1.8}$$

$$\partial y/\partial\Psi = 1/\rho u \sin\theta \tag{1.9}$$

$$v = u\,\partial y/\partial\theta \tag{1.10}$$

$$\rho u\,\partial h/\partial\theta = -\Gamma Q \tag{1.11}$$

On the body we assume the dimensionless stream function $\Psi = 0$, on the shock wave $\Psi = \Psi_S(\theta) = 1/2 \sin^2\theta$.

The boundary conditions (1.4) in new variables have the same form, but they must be taken for $\Psi = \Psi_S(\theta)$.

2. The solution of Eq. (1.7) with boundary conditions (1.4) has the form

$$u = \sqrt{2\Psi} \tag{2.1}$$

Taking account of this solution, the expression for the pressure from Eq. (1.8) with account of (1.4) takes the form

$$p(\Psi, \theta) = \cos^2\theta - \frac{1}{3}\sin^2\theta + \frac{(2\Psi)^{3/2}}{3\sin\theta} \tag{2.2}$$

For the geometrical coordinate we obtain the expression

$$y(\Psi, \theta) = \frac{1}{\sin\theta}\int\limits_0^\Psi \frac{d\Psi}{\rho\sqrt{2\Psi}} \tag{2.3}$$

Equation (1.10) determines the normal velocity component, which satisfies the boundary condition (1.4). For the final solution of the gasdynamic problem we must solve Eq. (1.11) with boundary condition (1.4). In the three-dimensional radiative-loss approximation the divergence of the radiant flow has the form (in dimensional form) [7-9]

$$Q = 2\mu\sigma T^4 \tag{2.4}$$

Here μ is the average Planck absorption coefficient. Analysis of tabular data [8-10] for air shows that in a definite range of pressures and temperatures the Planck coefficient can be approximated as

$$\mu(p, T) = ApT^n \tag{2.5}$$

where A and n are constants of the approximation, p is pressure, and T is temperature.

Using Eqs. (2.4) and (2.5) we obtain an equation for the enthalpy

$$u\frac{\partial h}{\partial \theta} = -\frac{bh^{n+5}}{(n+4)} \tag{2.6}$$

with the boundary condition

$$h(\Psi = \Psi_S(\theta)) = \cos^2\theta \tag{2.7}$$

where the dimensionless parameter b is the product of the characteristic optical thickness of the shock layer and the radiation parameter Γ

$$b = 2A\rho_\infty V_\infty{}^2\left(\frac{V_\infty{}^2}{2C_p}\right)^n \varepsilon R\frac{\gamma+1}{2\gamma}\Gamma(n+4) \tag{2.8}$$

In [4] this parameter is called the parameter of energy loss by radiation. The solution for the enthalpy has the form

$$h(\Psi, \theta) = \left\{\frac{1}{(1-2\Psi)^{n+4}} + \frac{b\theta}{\sqrt{2\Psi}} - \frac{b\arcsin\sqrt{2\Psi}}{\sqrt{2\Psi}}\right\}^\alpha, \quad \alpha = (n+4)^{-1} \tag{2.9}$$

Equation (2.9) enables us to find the density ρ, then the geometrical coordinate $y(\Psi, \theta)$ from Eq. (2.3) and the normal velocity component $v(\Psi, \theta)$ from Eq. (1.10).

Thus, we have determined all the parameters of the gas flow in the shock layer.

3. In most of the studies for the calculation of the radiant thermal flux on a body under the assumption that the shock layer is thin in comparison with the radius of the body, an equation for the plane layer with thickness equal to the distance that the shock wave has withdrawn is used. If it is assumed that the degree of blackness of the body equals unity, and the self-radiation of a comparatively cold surface of a body can be neglected, then the radiant flux on the body from the gas layer of each configuration is expressed by the equation of [10]. From this general expression, assuming that the shock layer is thin and that there is a three-dimensional radiative loss, we can obtain the following expression for the radial radiant flux (in dimensional form) at the point determined by the coordinate θ on the surface of the body

$$q_R(\theta) = \varepsilon R\int_0^{y_S(\theta)} dy\,\mu(p, T)\,\sigma T^4 \tag{3.1}$$

where θ is the angle determining the observation point on the body, $y_S(\theta)$ is the withdrawal of the shock wave.

Equation (3.1) can be rewritten in dimensionless form, using (2.8) and (2.9)

$$\frac{2q_R(\theta)}{\rho_\infty V_\infty{}^3} = \frac{b}{2(n+4)}\int_0^1\left\{(1-t^2\sin^2\theta)^{-(n+4)} + \frac{b(\theta - \arcsin(t\sin\theta))}{t\sin\theta}\right\}^m dt$$

$$m = -(n+5)/n+4 \tag{3.2}$$

This expression implies a number of special cases. For $b \ll 1$, when the radiation can be considered as a perturbation superimposed on an adiabatic gas flow, Eq. (3.2) yields

$$\frac{2q_R(\theta)}{\rho_\infty V_\infty{}^3} = \frac{b}{2(n+4)}\int_0^1 dt\,[1 - t^2\sin^2\theta]^{n+5} \tag{3.3}$$

For angles that satisfy the condition

$$\sin\theta < \sqrt{3/(n+5)}$$

the integral in (3.3) is calculated, retaining two terms in the binomial expansion of the integrand. Approximating the result obtained with the same accuracy, we obtain

$$\frac{2q_R(\theta)}{\rho_\infty V_\infty{}^3} = \frac{b}{2(n+4)}(\cos\theta)^{2/3(n+5)} \tag{3.4}$$

From this equation we see that the radiant flux increases in proportion to the radius of the body (or the thickness of the compressed layer) and decreases quite rapidly with increasing angle. The ratio of flow (3.4) to the flow at the critical point ($\theta = 0$) is independent of the parameter b and, hence, the radius of the body. These conclusions are confirmed by numerical calculations [4-6]. An approximation of the tabular data for air [9, 10] for $0.1 \leq p \leq 1$ atm, $4000°K \leq T \leq 11000°K$ gives the value $n \approx 8.0$. Equation (3.4) for $n=8$ leads to the angular dependence

$$q_R(\theta) \sim (\cos\theta)^{8.67}$$

Numerical calculations of the radiant flux for air for an incident-flow velocity $V_\infty = 10$ km/sec, when the gas can be assumed to be weakly radiating, leads to the dependence

$$q_R(\theta) \sim (\cos\theta)^{10.7}$$

In the other limiting case, when the gas is strongly radiating ($b \gg 1$), an asymptotic calculation of the integral in (3.2) leads to the following equation for the flow:

$$2q_R(\theta)/\rho_\infty V_\infty^3 = \cos^3\theta/2 \tag{3.5}$$

From this equation we see that for a strongly radiating gas the limiting radiant flux on a body equals half the kinetic flux of the incident gas (the other half is lost by irradiation to the side of the shock wave).

Numerical calculations [6] for $V_\infty = 16$ km/sec lead to the dependence $q_R(\theta) \sim \cos^{3.05}\theta$, which is close to the limiting expression (3.5).

From (3.5) it follows that for strong radiative loss, the radiant flux is independent of the radius of the body. This result is confirmed by numerical calculation [5] for the critical point, where it is shown that the ratio $q_R(R)/q_R(R=3\text{ m})$ approaches a constant limit for $R > 1$ m.

In both limiting cases (3.4) and (3.5) the ratio of the radiant flux for $\theta \neq 0$ to the flux at the critical point is independent of the parameter b and, hence, the radius of the body. The lack of dependence of this ratio on the radius is confirmed by numerical calculations [4, 6].

4. From the relations (2.3) and (2.9) given above, we can obtain the dimensionless withdrawal of the shock wave $y_S(\theta)$ as a function of the angle θ and the parameter b

$$y_S(\theta) = \frac{\gamma+1}{2\gamma}\int_0^1 dt\left[\cos^2\theta - \frac{1}{3}\sin^2\theta + \frac{t^3}{3}\sin^2\theta\right]^{-1}\left\{\frac{1}{(1-t^2\sin^2\theta)^{n+4}} + \frac{b(\theta - \arcsin(t\sin\theta))}{t\sin\theta}\right\}^{-1/(n+4)} \tag{4.1}$$

For a weakly radiating gas for $b \ll 1$, Eq. (4.1) implies

$$y_S(\theta) = \frac{\gamma+1}{2\gamma}\int_0^1 \frac{dt\,(1-t^2\sin^2\theta)}{[\cos^2\theta - {}^1/_3\sin^2\theta\,(1-t^3)]} \tag{4.2}$$

Calculation of the integral in (4.2) gives

$$y_S(\theta) = \frac{\gamma+1}{2\gamma}\left\{\frac{3}{a^2\sin^2\theta}\left[\frac{1}{6}\ln\frac{(a+1)^2}{a^2-a+1} + \right.\right.$$
$$\left.\left. + \frac{1}{\sqrt{3}}\left(\operatorname{arc\,tg}\frac{2-a}{a\sqrt{3}} + \operatorname{arc\,tg}\frac{1}{\sqrt{3}}\right)\right] - \ln\frac{\cos^2\theta}{\cos^2\theta - {}^1/_3\sin^2\theta}\right\} \tag{4.3}$$

where $a^3 = 3\cot^2\theta - 1$.

For angles $\theta < \pi/6$, Eq. (4.2) gives, with accuracy up to 12%,

$$y_S(\theta) = \frac{\gamma+1}{2\gamma}\frac{1}{\cos^2\theta} \tag{4.4}$$

Thus, for a weakly radiating gas, the withdrawal naturally is independent of the parameter b, i.e., independent of the radiation, and increases with increasing angle θ.

For a strongly radiating gas with $b \gg 1$ an asymptotic calculation of the integral in (4.1) for angles $\theta < \pi/6$ gives

$$y_S(\theta) = \frac{\gamma+1}{2\gamma}\frac{n+4}{n+3}\frac{b^\alpha}{\cos^2\theta} \tag{4.5}$$

The withdrawal of the shock wave in dimensional form

$$r_S(\theta) - R = \varepsilon R y_S(\theta) \tag{4.6}$$

where $r_S = r_S(\theta)$ is the equation that describes the form of the shock wave. From Eqs. (4.3), (4.5), and (4.6) it follows that for a weakly radiating gas the withdrawal increases $\sim R$, and for a strongly radiating gas $\sim R^{(n+3)/(n+4)}$, and decreases with increasing velocity V_∞. This is confirmed by an analysis of the numerical studies [5, 7]. It is interesting to note that Eqs. (4.4) and (4.5) imply the same dependence of the withdrawal on angle θ for strongly and weakly radiating gas, at least for angles $\theta < \pi/6$. The ratio of the withdrawal for $\theta \neq 0$ to the withdrawal at the critical line $\theta = 0$ in both limiting cases is independent of parameter b, which is related to the radiation.

Equations (4.1) and (4.5) imply that the withdrawal of the shock wave decreases with increasing parameter of energy loss by radiation b. Such a decrease in withdrawal of the shock wave for a three-dimensional radiative loss was noted in the calculations of [7].

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